Here are two computational examples of Lie derivatives $\mathcal{L}_X Y$.

① $\mathbb{R}^2 \setminus \{0\}$ in polar coordinate (ρ, θ)

$$X(\rho,\theta) = (0,1) \Rightarrow \varphi_X^t(\rho,\theta) = (\rho, \theta+t)$$

recall $\varphi_X^t(\rho,\theta)$ is defined by moving along the flowlines of X.

$$Y(\rho,\theta) = (\rho, 0)$$

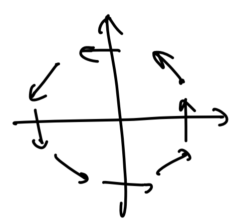
then

$$(\mathcal{L}_X Y)(\rho_0,\theta_0) = \lim_{t\to 0} \frac{(\varphi_X^{-t})_*(\varphi_X^t(\rho_0,\theta_0))\, Y(\varphi_X^t(\rho_0,\theta_0)) - Y(\rho_0,\theta_0)}{t}$$

$$= \lim_{t\to 0} \frac{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \rho_0 \\ 0 \end{pmatrix} - \begin{pmatrix} \rho_0 \\ 0 \end{pmatrix}}{t} = 0$$

② $\mathbb{R}^2$ in coordinate (x,y)

$$X(x,y) = (-y, x) \Rightarrow$$



linear map

↓

$$\Rightarrow \varphi_X^t(x,y) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$Y(x,y) = (1,0)$$

Then

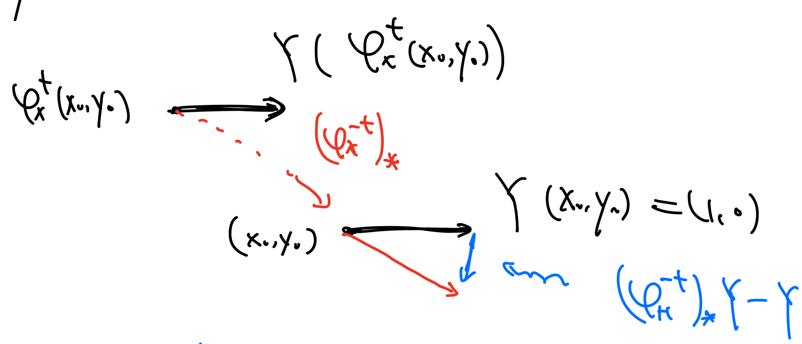
$$(\mathcal{L}_X Y)(x_0,y_0) = \lim_{t\to 0} \frac{(\varphi_X^{-t})_*(\varphi_X^t(x_0,y_0))\, Y(\varphi_X^t(x_0,y_0)) - Y(x_0,y_0)}{t}$$

$$= \lim_{t\to 0} \frac{\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix}}{t}$$

$$= \lim_{t\to 0} \frac{\begin{pmatrix} \cos t - 1 \\ -\sin t \end{pmatrix}}{t}$$

$$= \begin{pmatrix} \lim_{t\to 0} \frac{\cos t - 1}{t} \\ \lim_{t\to 0} \frac{-\sin t}{t} \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

Geometrically



when $t\to 0$, vector ↙ approximates to $(0,-1)$.

Rmk Compute $[X,Y]$.

$$[X,Y] = \left(D_X Y^1 - D_Y X^1,\ D_X Y^2 - D_Y X^2 \right)$$

$$= \left(0 - (0,-1)\begin{pmatrix} 1 \\ 0 \end{pmatrix},\ 0 - (1,0)\begin{pmatrix} 1 \\ 0 \end{pmatrix} \right)$$

$$= (0,-1)$$

Note that $L_X Y = [X,Y]$. This is <u>not</u> a coincidence!

Prop (Exe). $L_X Y = [X,Y]$

Therefore, when input is a vector field, $L_X(-)$ does <u>not</u> provide any new info (in terms of computations).